Chapter 2 Stress and Strain

2.0 INTRODUCTION
In the first Chapter we discuss the equations of statics, and how to determine the ground reaction for any structure. The method can also be used to determine the internal loads carried by the members or parts of a body. We now need to define how these internal loads are distributed and carried by the material and the deformation they create.

2.1 STRESS (SI&4th p. 22-23, 3rd p. 22-23)
Consider an element of continuous (no voids) and cohesive (no cracks, breaks and defects) material subjected to a number of externally applied loads as shown in Fig. 2.1a). It is supposed that the member is in equilibrium.

![Free Body Diagram](image)

Fig. 2.1 External and internal forces in a structural member

If we now cut this body, the applied forces can be thought of as being distributed over the cut area A as in Fig. 2.1b). Now if we look at infinitesimal regions $\Delta A$, we assume the resultant force in this infinitesimal area is $\Delta F$. In fact, $\Delta F$ is also a distributed force. When $\Delta A$ is extremely small, we can say that the distributed force $\Delta F$ is uniform. In other words, if we look at the whole sectioned area, we can say that the entire area A is subject to an infinite number of forces, where each one (of magnitude $\Delta F$) acts over a small area of size $\Delta A$. Now, we can define stress.

Definition: **Stress is the intensity of the internal force on a specific plane passing through a point.**

Mathematically, stress can be expressed as

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$  \hspace{1cm} (2.1)

Dividing the magnitude of internal force $\Delta F$ by the acting area $\Delta A$, we obtain the stress. If we let $\Delta A$ approach zero, we obtain the stress at a point. In general, the stress could vary in the body, which depends on the position that we are concerning. The stress is one of most important concepts that we introduced in mechanics of solids.

**Normal and Shear Stress**
As we known, force is a vector that has both magnitude and direction. But in the stress definition, we only consider the magnitude of the force. Obviously, this may easily confuse us. Let’s still take patch $\Delta A$ as an example. As we can see, force $\Delta F$ is not perpendicular to the sectioned infinitesimal area $\Delta A$. If we only take the magnitude of the force into account,
apparently, the stress may not reflect the real mechanical status at this point. In other words, we need to consider both magnitude and direction of the force.

Now let’s resolve the force $\Delta F$ in \textit{normal} and \textit{tangential} direction of the acting area as Fig. 2.1b). The intensity of the force or force per unit area acting normally to section $A$ is called \textbf{Normal Stress}, $\sigma$ (sigma), and it is expressed as:

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A}$$ \hspace{1cm} (2.2)

If this stress “pulls” on the area it is referred as \textbf{Tensile Stress} and defined as \textit{Positive}. If it “pushes” on the area it is called \textbf{Compressive Stress} and defined as \textit{Negative}.

The intensity or force per unit area acting tangentially to $A$ is called \textbf{Shear Stress}, $\tau$ (tau), and it is expressed as:

$$\tau = \lim_{\Delta A \to 0} \frac{\Delta F_t}{\Delta A}$$ \hspace{1cm} (2.3)

\textbf{Average Normal Stress} (SI&4\textsuperscript{th} p. 24-31, 3\textsuperscript{rd} p. 27-34)

To begin, we only look at beams that carry tensile or compressive loads and which are long and slender. Such beams can then be assumed to carry a constant stress, and Eq. (2.2) can be simplified to:

$$\sigma = \frac{F}{A}$$ \hspace{1cm} (2.4)

We call this either \textbf{Average Normal Stress} or \textbf{Uniform Uniaxial Stress}.

\textbf{Units of Stress}

The units in the SI system is the Newton per square meter or \textbf{Pascal}, i.e.: $\text{Pa} = \text{N/m}^2$.

In engineering, Pa seems too small, so we usually use:\

- \textbf{Kilo Pascal} (KPa) (=Pa$x\times10^3$) e.g. 20,000Pa=20kPa
- \textbf{Mega Pascal} (MPa) (=Pa$x\times10^6$) e.g. 20,000,000Pa=20MPa
- \textbf{Giga Pascal} (GPa) (=Pa$x\times10^9$) e.g. 20,000,000,000Pa=20GPa

\textbf{Example 2.1}: An 80 kg lamp is supported by a single electrical copper cable of diameter $d = 3.15$ mm. What is the stress carried by the cable.

To determine the stress in the wire/cable as Eq. (2.4), we need the cross sectional area $A$ of the cable and the applied internal force $F$:

$$A = \frac{\pi d^2}{4} = \frac{\pi \times 0.00315^2}{4} = 7.793 \times 10^{-6} \text{ m}^2$$

$$F = mg = 80 \times 9.8 = 784 \text{ N}$$

so

$$\sigma = \frac{F}{A} = \frac{784}{7.793 \times 10^{-6}} = 100.6 \text{ MPa}$$

\textbf{Allowable Stress} (SI&4\textsuperscript{th} p. 48-49, 3\textsuperscript{rd} p. 51-52)

From Example 2.1, we may concern whether or not 80kg would be too heavy, or say 100.6 MPa stress would be too high for the wire/cable, from the safety point of view. Indeed, stress is one of most important indicators of structural strength. When the stress (intensity of force) of an element exceeds some level, the structure will fail. For convenience, we usually adopt \textit{allowable force} or \textit{allowable stress} to measure the threshold of safety in engineering.
Moreover, there are following several reasons that we must take into account in engineering:

- The load for design may be different from the actual load.
- Size of structural member may not be very precise due to manufacturing and assembly.
- Various defects in material due to manufacturing processing.

One simple method to consider such uncertainties is to use a number called the \textbf{Factor of Safety, F.S.}, which is a ratio of failure load $F_{\text{fail}}$ (found from experimental testing) divided by the allowable one $F_{\text{allow}}$

$$F.S. = \frac{F_{\text{fail}}}{F_{\text{allow}}} \quad (2.5)$$

If the applied load is linearly related to the stress developed in the member, as in the case of using $\sigma = F/A$, then we can define the factor of safety as a ratio of the failure stress $\sigma_{\text{fail}}$ to the allowable stress $\sigma_{\text{allow}}$

$$F.S. = \frac{\sigma_{\text{fail}}}{\sigma_{\text{allow}}} \quad (2.6)$$

Usually, the factor of safety is chosen to be greater than 1 in order to avoid the potential failure. This is dependent on the specific design case. For nuclear power plant, the factor of safety for some of its components may be as high as 3. For an aircraft design, the higher the F.S. (safer), the heavier the structure, therefore the higher in the operational cost. So we need to balance the safety and cost.

The value of F.S. can be found in design codes and engineering handbook. More often, we use Eq. (2.6) to compute the allowable stress:

$$\sigma_{\text{allow}} = \frac{\sigma_{\text{fail}}}{F.S.} \quad (2.7)$$

\textbf{Example 2.2:} In Example 2.1, if the maximum allowable stress for copper is $\sigma_{\text{Cu,allow}}=50\text{MPa}$, please determine the minimum size of the wire/cable from the material strength point of view.

Mathematically,

$$\sigma = \frac{F}{A} = \frac{mg}{\pi d^2/4} \leq \sigma_{\text{Cu,allow}}$$

Therefore:

$$d \geq \sqrt{\frac{4mg}{\pi \sigma_{\text{Cu,allow}}}} = 4.469 \times 10^{-3} = 4.469 \text{mm}$$

Obviously, the lower the allowable stress, the bigger the cable size. Stress is an indication of structural strength and elemental size.

In engineering, there are two significant problems associated with stress as follows.

\textbf{Problem (1) Stress Analysis:} for a specific structure, we can determine the stress level. With the stress level, we then justify the safety and reliability of a structural member, i.e. known size $A$ and load $F$, to determine stress level: $\sigma = F/A$

\textbf{Problem (2) Engineering Design:} Inversely, we can design a structural member based on the allowable stress so that it can satisfy the safety requirements, i.e. known material’s allowable stress $\sigma_{\text{allow}}$ and load $F$, to design the element size: $A \geq F/\sigma_{\text{allow}}$
2.2 DEFORMATION (SI&4th p. 67-68, 3rd p. 70)
Whenever a force is applied to a body, its shape and size will change. These changes are referred as deformations. These deformations can be thought of being either positive (elongation) or negative (contraction) in sign as shown in Fig. 2.2.

![Fig. 2.2 Deformation due to applied axial forces](image)

It is however very hard to make a relative comparison between bodies or structures of different size and length as their individual deformations will be different. This requires the development of the concept of Strain, which relates the body’s deformation to its initial length.

2.3 STRAIN (SI p.68-69; &4th p. 67-69; 3rd p.71-72)

**Normal Strain**
The elongation (+ve) or contraction (−ve) of a body per unit length is termed Strain.

![Fig. 2.3 Generalized deformation due to applied forces](image)

Let’s take the arbitrarily shaped body in Fig. 2.3 as an example. Consider the infinitesimal line segment $AB$ that is contained within the undeformed body as shown in Fig. 2.3(a). The line $AB$ lies along the $n$-axis and has an original length of $\Delta S$. After deformation, points A and B are displaced to $A'$ and $B'$ and in general the line becomes a curve having a length $\Delta S'$. The change in length of the line is therefore $\Delta S'-\Delta S$. We consequently define the generalized strain mathematically as

$$\varepsilon = \lim_{B \to A} \frac{\Delta S' - \Delta S}{\Delta S} \quad (B \to A \text{ along } n)$$

(2.8)

**Average Normal Strain**
If the stress in the body is everywhere constant, in other words, the deformation is uniform in the material (e.g. uniform uniaxial tension or compression) as shown in Fig. 2.2, the strain can be computed by

$$\varepsilon = \frac{L_{\text{Deformed}} - L_{\text{Original}}}{L_{\text{Original}}} = \frac{\Delta L}{L}$$

(2.9)

i.e. the change in length of the body over its original length,
Unit of Strain
From Eqs. (2.8) and (2.9), we can notice that the normal strain is a dimensionless quantity since it is a ratio of two lengths. Although this is the case, it is common in practice to state it in terms of a ratio of length units, i.e. meters per meter (m/m).

Usually, for most engineering applications $\varepsilon$ is very small, so measurements of strain are in micrometers per meter ($\mu$m/m) or ($\mu$/m). Sometimes for experiment work, strain is expressed as a percent, e.g. $0.001m/m = 0.1\%$.

A normal strain of $480\mu$m for a one-meter length is said:

$$\varepsilon = 480\times10^{-6} = 480(\mu$m/m) = 0.0480\% = 480\mu$s (micro strain)

Example 2.3: In Example 2.1, if it is measured that the cable was elongated by $1.35\ mm$ due to the weight of the light, what would its strain be?

$$\varepsilon = \frac{0.00135}{1.5} = 900\times10^{-6} = 900\mu$s

2.4 STRESS-STRAIN RELATIONSHIP, HOOKE'S LAW (SI&4th p.83-93; 3rd p.85-95)
Material Test and Stress-Strain Diagram
The material strength depends on its ability to sustain a load without undue deformation or failure. The property is inherent in the material itself and must be determined by experiment. One of the most important tests to perform in this regard is the tension or compression. To do so, a bunch of standard specimen is made. The test is performed in universal test machine. Shown in Fig. 2.4 is the specimen and test result of Stress-Strain Diagram.

Fig. 2.4 Material test and Stress-Strain Diagram

The Stress-Strain diagram consists of 4 stages during the whole process, elastic, yielding, hardening and necking stages respectively. From yielding stage, some permanent plastic deformation occurs. About 90% of engineering problems only concern the elastic deformation in structural members and mechanical components. Only 10% of engineering work concerns plastic and other nonlinear stage (e.g. metal forming). In this subject, we are only involved in the linear elastic region, in which the relationship between the strain and stress is linear.

Hooke’s Law
The stress-strain linear relationship was discovered by Robert Hook in 1676 and is known as Hooke's law. It is mathematically represented by Eq. (2.10),
\[
\sigma = E\varepsilon \quad \text{(2.10)}
\]

where \( E \) is terms as the **Modulus of Elasticity** or **Young's Modulus** with units of N/m\(^2\) or Pa. For most of engineering metal material, GPa is used, e.g. mild steel is about 200GPa~210GPa.

### 2.5 POISSON'S RATIO (SI\&4\(^{th}\) Ed p104-105; 3\(^{rd}\) Ed p107-108)

**Definition**

When a deformable body is stretched by a tensile force, not only does it elongate but it also contract laterally, i.e. it would contract in other two dimensions as shown in Fig 2.5(a). Likewise, a compressive force acting on a deformable body cause it to contract in the direction of force and yet its sides expand laterally as in Fig. 2.5(b).

When the load \( P \) is applied to the bar it changes the bar’s length by \( \delta \) and its radius by \( \delta_r \). The strain in axial direction and in lateral/radial direction are respectively

\[
\begin{align*}
\varepsilon_{\text{axial}} &= \frac{\delta}{L} \\
\varepsilon_{\text{lateral}} &= \frac{\delta_r}{r}
\end{align*}
\]

In early 1800s, French scientist Poisson realized that within elastic range the ratio of these two strains is a constant. We called the constant as **Poisson’s ratio** by \( \nu \) (Nu)

\[
\nu = -\frac{\text{Lateral Strain}}{\text{Axial Strain}} = -\frac{\varepsilon_{\text{lateral}}}{\varepsilon_{\text{axial}}} \quad \text{(2.11)}
\]

![Fig. 2.5 Relationship of the axial strain with the lateral strain](image)

The *negative sign* is used here since longitudinal elongation (positive strain) cause lateral contraction (negative strain), *vice versa*. So Poisson’s ration is positive, i.e. \( \nu \geq 0 \).

**Remarks**

- The lateral strain is caused only by axial force. No force or stress acts in lateral direction;
- Lateral strain is the same in all lateral direction;
- Usually \( 0 \leq \nu \leq 0.5 \). For most linearly elastic material \( \nu = 0.3 \);
- Poisson’s ratio is a constant.

**Strain in Lateral Direction**

For bars subjected to a tensile stress \( \sigma_x \), the strains in the \( y \) and \( z \) planes are:
2.6 THERMAL STRAIN (SI&4th Ed p. 148-152; 3rd Ed p. 152-156)

Thermal Deformation

When the temperature of a body is changed, its overall size will also change. In other words, temperature change may cause the dimension or shape change in the material. More specially, if the temperature increases, generally a material expands. Whereas if the temperature decreases, the material will contract. It is supposed that this is a common sense for anyone.

For the majority of engineering materials this relationship is linear. If we assume that the material is homogeneous and isotropic, from experiment, we can find a linear relation between thermal deformation and temperature change as:

\[
\delta_T = \alpha \cdot \Delta T \cdot L
\]  

where: \( \alpha \) : Coefficient of thermal expansion, units are strain per \( ^\circ \text{C} \)

\( \Delta T \): algebraic change in temperature (\( ^\circ \text{C} \)) (increase +; decrease −)

\( \delta_T \): algebraic change in length (“+” = elongation; “−” = contraction)

Thermal Strain

\[
\varepsilon_{\text{thermal}} = \frac{\delta_T}{L} = \alpha \cdot \Delta T
\]  

Coupled Strain Status

If we consider both mechanical strain \( \varepsilon_m \) and thermal strain \( \varepsilon_T \) in the structure as shown in Fig. 2.6(b), by referring to Eq. (2.12), the total strains in all directions would be computed as:

\[
\begin{align*}
\varepsilon_x &= \varepsilon_T + \varepsilon_{\sigma_x} = \alpha \Delta T + \frac{\sigma_x}{E} \\
\varepsilon_y &= \varepsilon_T + \varepsilon_{\sigma_y} = \alpha \Delta T - \frac{\sigma_y}{E} \\
\varepsilon_z &= \varepsilon_T + \varepsilon_{\sigma_z} = \alpha \Delta T - \frac{\sigma_z}{E}
\end{align*}
\]  

\[(2.15)\]
2.7 ELASTIC DEFORMATION OF AXIALLY LOADED MEMBER

(SI&4th p.120-127;3rd p.122-129)

Now we are going to find the elastic deformation of a member subjected to axial loads. Let’s consider a generalized bar shown in Fig. 2.7, which has a gradually varying cross-sectional area along its length \( L \). For a more general case, the bar is subjected to concentrated loads at its right end and also a variable external load distributed along its length (such as a distributed load could be for example, to represent the weight of a vertical bar or friction forces acting on bar surface). Here we wish to find the relative displacement \( \delta \) of one end with respect to the other.

![Fig. 2.7 Thermal and mechanical deformation](image)

We pick a differential element of length \( dx \) and cross-sectional area \( A(x) \). FBD can be drawn as middle of Fig. 2.7. Assume that resultant internal axial force is represented as \( P(x) \). The load \( P(x) \) will deform the element into the shape indicated by the dashed outline.

The average stress in the cross-sectional area would be \( \sigma(x) = \frac{P(x)}{A(x)} \)

The average strain in the cross-sectional area would be \( \varepsilon(x) = \frac{d\delta}{dx} \)

Provided these quantities do not exceed the proportional limit, we can relate them using Hook’s law, i.e. \( \sigma = E\varepsilon \)

Therefore

\[
\frac{P(x)}{A(x)} = E(x)\left(\frac{d\delta}{dx}\right)
\]

Re-organize the equation, we have

\[
d\delta = \frac{P(x)}{A(x)E(x)}dx
\]

For the entire length \( L \) of the bar, we must integrate this expression to find the required end displacement

\[
\delta = \int_0^L \frac{P(x)}{A(x)E(x)} dx \quad (2.16)
\]

Where:
\( \delta \) = displacement between two points
\( L \) = distance between the points
\( P(x) \) = Internal axial force distribution
\( A(x) \) = Cross-sectional area
\( E(x) \) = Young’s modulus
**Constant Load and Cross-Sectional Area**

In many engineering cases, the structural members experience a constant load and have a constant cross-sectional area and made of one homogenous material, i.e.

- \( P(x) = P = \text{constant} \) (no axially distributed load)
- \( A(x) = A = \text{constant} \) (uniform area)
- \( E(x) = E = \text{constant} \) (homogeneous material)

From Eq. (2.16), we have

\[
\delta = \frac{PL}{EA} \quad (2.17)
\]

**Multi-Segment Bar**

If the bar is subjected to several different axial forces or cross-sectional areas or Young’s moduli, the above equation can be used for each segment. The total displacement can be computed from algebraic addition as

\[
\delta = \sum_i \frac{P_i L_i}{A_i E_i} \quad (2.18)
\]

**Example 2.4:** The composite bar shown in the figure is made of two segments, AB and BC, having cross-sectional areas of \( A_{AB} = 200 \text{mm}^2 \) and \( A_{BC} = 100 \text{mm}^2 \). Their Young’s moduli are \( E_{AB} = 100 \text{GPa} \) and \( E_{BC} = 210 \text{GPa} \) respectively. Find the total displacement at the right end.

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Step 1 FBDs for Segments AB and BC.
Assume the internal forces are in tension.

Step 2 Equilibriums
Internal force in AB

\[
\sum F_x = 0 = -P_{AB} - F_2 + F_1 = 0
\]

\[
\therefore P_{AB} = -30 \text{kN}
\]

(Opposite to our assumption of tension, so Segment AB is in compression)

Internal force in BC

\[
\sum F_x = 0 = -P_{BC} + F_1 = 0
\]

\[
\therefore P_{BC} = F_1 = 10 \text{kN} \quad \text{(in tension)}
\]

Step 3 Compute the total deformation by using Eq. (2.18)

\[
\delta_{AC} = \delta_{AB} + \delta_{BC} = \frac{P_{AB} L_{AB}}{E_{AB} A_{AB}} + \frac{P_{BC} L_{BC}}{E_{BC} A_{BC}} = \frac{-30 \times 10^3 \times 4}{100 \times 10^9 \times 200 \times 10^{-6}} + \frac{10 \times 10^3 \times 4.2}{210 \times 10^9 \times 100 \times 10^{-6}}
\]

\[
\delta_{AC} = -0.006 + 0.002 = -0.004 \text{m} = -4 \text{mm} \quad \text{(towards left)}
\]
2.8 STATICALLY INDETERMINATE MEMBERS LOADED AXIALLY
(Sl&4th Ed p. 134-139; 3rd Ed p. 137-142)

Statically Determinate and Indeterminate

When a bar is supported at one end and subjected to an axial force $P$ at the other end as shown in Fig. 2.8(a), there is only one unknown reaction force $F_A$. By using the equations of statics, the unknown reaction can easily be determined. So such a system with the same number of unknown reactions as equations of statics is called *statically determinate*. – i.e. known reactions can be determined strictly from equilibrium equations.

![Fig. 2.8 Statically determinate and indeterminate structures](image)

If the bar is also restricted at the free end as shown in Fig. 2.8(b), it has 2 unknown reactions $F_A$ and $F_B$, one known force $P$ and one equation of statics as:

$$ + \sum F_y = 0 = F_A + F_B - P = 0 \quad \therefore F_A + F_B = P$$

(2.19)

It cannot be solved if do not introduce more other conditions. If the system has more unknown forces than equations of statics it is called *statically indeterminate*.

Compatibility Conditions

What we need is an additional equation that specifies how the structure is displaced due to the applied loading. Such an equation is usually termed the *compatibility equation*.

Since there are 2 unknown and only 1 equation of statics herein, what we need is an additional equation that specifies how the structure is displaced due to the applied loading. Such an equation is usually termed the *compatibility equation or kinematic conditions*.

In order to determine the compatibility for this example we need to determine how point C is going to move, and how much point B moves in relation to point A. Now, since both ends of the bar are fully fixed, then the total change in length between A and B must be zero.

Basically the amount that length AC elongates CB contracts as shown in Fig. 2.9, so the equation can be written as:

$$ \delta_{AC} + \delta_{CB} = 0 $$

(2.20)
Let’s respectively look at the free body diagram for segment AC and CB as in Fig. 2.9. (indeed FBD can be in any level of structural system or structural members).

For segment AC,
\[ + \sum F_y = 0 = F_A - P_{AC} = 0 \quad \therefore P_{AC} = F_A \quad \text{Tension (+)} \]
\[ \delta_{AC} = \frac{P_{AC}L_{AC}}{A_{AC}E_{AC}} = \frac{F_AL_{AC}}{A_{AC}E_{AC}} \quad \text{elongation (+)} \quad (2.21) \]

For segment CB,
\[ + \sum F_y = 0 = F_B + P_{CB} = 0 \quad \therefore P_{CB} = -F_B \quad \text{Compression (−)} \]
\[ \delta_{CB} = -\frac{P_{CB}L_{CB}}{A_{CB}E_{CB}} = -\frac{F_BL_{CB}}{A_{CB}E_{CB}} \quad \text{contraction (−)} \quad (2.22) \]

Compatibility condition:
\[ \delta = \delta_{AC} + \delta_{CB} = \frac{F_AL_{AC}}{A_{AC}E_{AC}} + \left( -\frac{F_BL_{CB}}{A_{CB}E_{CB}} \right) = 0 \quad (2.23) \]

Combining Compatibility equation (2.23) with the equation of statics (2.19), we now can solve for the two unknowns \( F_A \) and \( F_B \) as,
\[ \begin{cases} \frac{F_AL_{AC}}{A_{AC}E_{AC}} - \frac{F_BL_{CB}}{A_{CB}E_{CB}} = 0 \\ F_A + F_B = P \end{cases} \quad (2.24) \]

i.e.
\[ F_B = \frac{A_{AC}E_{AC}}{A_{AC}E_{AC} + A_{CB}E_{CB}} P \quad F_A = \frac{A_{CB}E_{CB}}{A_{AC}E_{AC} + A_{CB}E_{CB}} P \]

If \( A_{AC}E_{AC} = A_{CB}E_{CB} = \text{Const} \), we have
\[ F_B = \frac{L_{AC}}{L} P \quad \text{and} \quad F_A = \frac{L_{CB}}{L} P \quad (2.25) \]
Example 2.5: Two bars made of Copper and Aluminum are fixed to the rigid abutments. Originally, there is a gap of 5mm between the ends as shown in the figure. Determine average normal stress in both bars if increase the temperature from 10°C to 210°C.

\[ \alpha_{cu} = 17 \times 10^{-6} \]
\[ E_{cu} = 110 \text{GPa} \]
\[ d = 0.01 \text{m} \]

\[ \alpha_{al} = 23 \times 10^{-6} \]
\[ E_{al} = 69 \text{GPa} \]
\[ \delta_{T,cu} \quad \text{Thermally expanded} \quad \delta_{F,cu} \quad \text{due to} \quad \Delta T \]

\[ \Delta T = 210 - 10 = 200^\circ C \]
\[ T_0 = 10^\circ C \]
\[ T = 210^\circ C \]
\[ d = 0.01 \text{m} \]
\[ \delta_{T,cu} \quad \delta_{F,cu} \]

\[ A = \frac{\pi}{4} d^2 = \frac{3.14}{4} 0.01^2 = 7.85 \times 10^{-5} \text{m}^2 \]
\[ \Delta T = 210 - 10 = 200^\circ C \]

Let’s firstly look at the copper bar. When the bar system is heated up from 10°C to 210°C, the copper bar expand towards right by \( \delta_{T,cu} \). After the copper bar touch to the aluminum bar, a mechanical force \( F \) will develop, which will prevent the copper bar from expanding further. We assume that due to such a mechanical force, the copper bar is pressed back by \( \delta_{F,cu} \). The real total deformation of copper bar will be computed as

\[ \delta_{Cu} = \delta_{T,cu} - \delta_{F,cu} \quad \text{(elongation +, Contraction -)} \]

Similarly, we have

\[ \delta_{Al} = \delta_{T,Al} - \delta_{F,Al} \quad \text{(elongation +, Contraction -)} \]

Because these two expanding bars should fill the gap, we prescribe a compatibility condition as

\[ \delta_{Cu} + \delta_{Al} = 0.005 \quad (2.26) \]

From these two equations, we have

\[ (\delta_{T,cu} - \delta_{F,cu}) + (\delta_{T,Al} - \delta_{F,Al}) = 0.005 \]

i.e.

\[ \left( \alpha_{Cu} \Delta T \frac{L_{Cu}}{E_{Cu} A_{Cu}} - \frac{F \times L_{Cu}}{E_{Cu} A_{Cu}} \right) + \left( \alpha_{Al} \Delta T \frac{L_{Al}}{E_{Al} A_{Al}} - \frac{F \times L_{Al}}{E_{Al} A_{Al}} \right) = 0.005 \]

\[ F = \frac{\alpha_{Cu} \Delta T L_{Cu} \frac{L_{Cu}}{E_{Cu} A_{Cu}} + \alpha_{Al} \Delta T L_{Al} \frac{L_{Al}}{E_{Al} A_{Al}} - 0.005}{0.4 + 0.8} = 17 \times 10^{-6} \times 200 \times 0.4 + 23 \times 10^{-6} \times 200 \times 0.8 - 0.005 \]

\[ = 206.2 \text{N} \]

The average normal stress can be computed as

\[ \sigma = \frac{F}{A} = \frac{206.2}{7.85 \times 10^{-5}} = 2.63 \text{MPa} \]
$$\delta_{Cu} = \delta_{T,Cu} - \delta_{F,Cu} = \left( \alpha_{Cu} \Delta T \frac{L_{Cu}}{E_{Cu} A_{Cu}} \right) = 17 \times 10^{-6} \times 200 \times 0.4 - \frac{206.2 \times 0.4}{110 \times 10^9 \times 7.85 \times 10^{-5}} = 1.36 \times 10^{-3} - 9.55 \times 10^{-6} = 1.35 \times 10^{-3} m = 1.35 mm \rightarrow$$

$$\delta_{Al} = \delta_{T,Al} - \delta_{F,Al} = \left( \alpha_{Al} \Delta T \frac{L_{Al}}{E_{Al} A_{Al}} \right) = 23 \times 10^{-6} \times 200 \times 0.8 - \frac{206.2 \times 0.8}{69 \times 10^9 \times 7.85 \times 10^{-5}} = 3.68 \times 10^{-3} - 3.05 \times 10^{-6} = 3.65 \times 10^{-3} m = 3.65 mm \leftarrow$$

### 2.9 AVERAGE SHEAR STRESS (SI&4th Ed p. 32-39; 3rd Ed p. 35-41)

The intensity or force per unit area acting tangentially to $A$ is called **Shear Stress**, $(\tau)$, and it is expressed as in Eq. (2.3) as:

$$\tau = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \quad (2.3)$$

In order to show how the shear stress can develop in a structural member, let’s take a block as an example. The block is supported by two rigid bodies and an external force $F$ is applied vertically as shown in Fig. 2.10. If the force is large enough, it will cause the material of the block to deform and fail along the vertical planes as shown.

A FBD of the unsupported center segment indicate that shear force $V = F/2$ must be applied at each section to hold the segment in equilibrium.

![Fig. 2.10 Average shear stress](image)

$$+ \sum F_y = 0 = 2V - F = 0$$

$$\therefore V = \frac{F}{2}$$

The average shear stress distributed over each sectioned area that develops the shear force is defined by

$$\tau_{avg} = \frac{V}{A} \quad (2.27)$$

$\tau_{avg}$ = assume to be the same at each point over the section

$V$ = Internal shear force

$A$ = Area at the section
2.10 STRESS CONCENTRATIONS (SI&\textsuperscript{4th} Ed p. 156-161; 3\textsuperscript{rd} Ed p. 159-163)

For a uniform cross-sectional bar that is applied an axial force, both experiment and theory of elasticity find that the normal stress will be uniformly distributed over the cross-section

**Stress Concentration**

However, if we drill a hole for some reasons in the component, the typical example is to build a connection with other structural elements. For such a case, if we cut from the hole’s center plane, we find that the stress distribution is no longer uniform, as in Fig. 2.11(a). It may distribute over such a smaller area in highly uneven pattern. We call this phenomenon as Stress Concentration.

**Stress Concentration Factor**

In engineering practice, though, the actual stress distribution does not have to be determined. Instead, only the maximum stress at these sections must be known, and the member is then designed to resist this highest stress when the axial load is applied. The specific values of the maximum normal stress at the critical section can be determined by experimental methods or by advanced mathematical techniques using the theory of elasticity. The results of these investigations are usually reported in graphical form (as in Fig. 2.11(c)) in terms of Stress Concentration Factor $K$.

\[
K = \frac{\sigma_{\text{max}}}{\sigma_{\text{avg}}} 
\]

in which $\sigma_{\text{avg}} = P/A'$ is the assumed average stress as in Fig. 2.11(b). Provided $K$ has been known from the figures or tables (as in Fig. 2.11(c)), and the average normal stress has been calculated from $\sigma_{\text{avg}} = P/A'$, where $A'$ is the smallest cross-sectional area. Then from the above equation the maximum stress at the cross section can be computed as:

\[
\sigma_{\text{max}} = K \sigma_{\text{avg}} = K \frac{P}{A'}
\]

Stress concentration occurs in the case that there is a sudden change in cross-sectional area. By observing Fig. 2.11(c), it is interesting to note that the bigger the ratio of change in the sectional area, the higher the stress concentration.