Topic 8: Circular Curves
Aims:

To differentiate between the different types of horizontal and circular curves

To understand the terminology and geometry of circular curves

To calculate through chainage values along the centre lines of circular curves

Design curves of constant radii to join straight section of for example a road or railway

Set out the centrelines of circular curves
Horizontal curves

In the design of roads or railways, straight sections of road or track are connected by curves of constant or varying radius as shown below:

The purpose of these curves is to deflect a vehicle travelling along one of the straights safely and comfortably through a deflection angle $\theta$ to enable it to continue its journey along the other straight. The two main types shown above are:
Circular curves, curves of constant radius.

Transition curves, curves of varying radius.

A road or railway will usually comprise of a series of straights, circular curves and transition curves, collectively known as the horizontal alignment.

Geometry of Circular Curves

There are 3 basic types of circular curves: simple curves; compound curves and reverse curves (all of which are also known as radius or degree curves)

Simple Circular Curves
A simple circular curve consists of one arc of constant radius R, these are the most commonly used type of curves (see previous fig part a).

Compound Circular Curves
These consist of two or more consecutive simple circular curves of different radii without and intervening straight section.
**Reverse Circular Curves**
These consist of two consecutive curves of the same or different radii with any intervening straight section and with their centres of curvature falling on opposite sides of their common tangent point ($T_C$).
Radius and Degree Curves
A circular curve can be referred to in one of two ways:

In terms of its radius e.g. a 750m curve, this is known as a radius curve.

In terms of the angle subtended at its centre by a 100m arc, for example a $2^\circ$ curve. (known as a degree curve)

Below, the arc VW=100m and it subtends an angle of $D^\circ$ at the centre of curvature O. TU is therefore a $D^\circ$ curve.

The relationship between radius curves and degree curves is given by:

$$DR = \frac{18,000}{\pi}, \text{ a 1500m radius curve is equivalent to } D^\circ = \frac{18,000}{1500\pi} = \frac{12}{\pi} = 3.820^\circ$$
Circular Curves Terminology
Although circular curves are quite simple the terminology used also applies to transition curves which are more complex. Thus it is important to understand all terms for circular curves first, before looking at transition curves.

- I is the intersection point of the two straights TI and IU
- TPU is a circular curve which runs around the arc from T to U
- The length of the circular curve around the arc TPU = L_c
- T and U are tangent points to the circular curve
- TI and IU are the tangent lengths of the circular curve
-P is the mid point of the circular curve TPU
-Long chord = TSU
-S is the mid point of the long chord TSU
-Deflection angle = \( \theta \) = external angle at I = angle CIU
-Intersection angle = \( 180^\circ - \theta \) = internal angle at I – TIU
-Radius of curvature of the curve = R
-Centre of curvature = O
-Q is any point on the circular curve TPU
-Tangential angle \( ITQ \) = the angle from the tangent length at T to and point on the circular curve
-The mid ordinate of the circular curve = PS
-Radius angle = angle TOU – deflection angle CIU = \( \theta \)
-External distance = PI

Formulae used in Circular Curves

Tangent lengths \( IT \) and \( IU \) (in triangle IUO):

\[
R \tan \left( \frac{\theta}{2} \right) = \frac{IU}{IO} = \frac{IU}{R} \quad \text{hence:} \quad IU = IT = R \tan \left( \frac{\theta}{2} \right)
\]
**External Distance, PI (in triangle IUO):**

\[
\cos\left(\frac{\theta}{2}\right) = \frac{R}{IO}
\]

Or

\[
IO = \frac{R}{\cos\left(\frac{\theta}{2}\right)} = R \sec\left(\frac{\theta}{2}\right), \text{ but } PI = OI - OP = OI - R
\]

Hence,

\[
PI = R \sec\left(\frac{\theta}{2}\right) - R = R \left[ \sec\left(\frac{\theta}{2}\right) - 1 \right]
\]

**Mid-ordinate, PS (in triangle TSO):**

\[
\cos\left(\frac{\theta}{2}\right) = \frac{OS}{OT} \quad \text{or} \quad OS = OT \cos\left(\frac{\theta}{2}\right) = R \cos\left(\frac{\theta}{2}\right)
\]

But

\[
PS = OP - OS = R - R \cos\left(\frac{\theta}{2}\right) = R \left[ 1 - \cos\left(\frac{\theta}{2}\right) \right]
\]
Long Chord, $TU$ (in triangle $USO$):

\[
\sin \left( \frac{\theta}{2} \right) = \frac{US}{UO} = \frac{US}{R} \quad \text{or} \quad US = R \sin \left( \frac{\theta}{2} \right)
\]

But $TU = US + TS$ and $US = TS$

Hence $TU = 2R \sin \left( \frac{\theta}{2} \right)$

Length of the Circular Curve $L_C$:

For a curve of radius $R$:

\[
L_C = R \theta \quad \text{metres}
\]

For a $D^\circ$ degree curve:

\[
L_C = 100 \left( \frac{\theta}{D} \right) \quad \text{metres}
\]

$R$ is in metres and $\theta$ is in radians in both cases.
Important relationships for Circular Curves for Setting Out

The triangle ITU is an isosceles triangle and therefore the angle ITU = IUT = (θ/2). The following definition can be given:

*The tangential angle α at T to any point X on the curve TU is equal to half the angle subtended at the centre of curvature O by the chord from T to that point.*

Similarly with reference to the figure below the following definition applies of a second tangent point:
The tangential angle $\beta$ at any point $X$ on the curve to any forward point $Y$ on the curve is equal to half the angle subtended at the centre by the chord between the two points.
The tangential angle to any point on the curve is equal to the sum of the tangential angles from each chord up to that point.

I.e. $\text{TOY} = 2(\alpha + \beta)$ and it follows that $\text{ITY} = (\alpha + \beta)$.

**Chainage**

Chainage is simply the longitudinal distance (usually in m) along a centreline from a start or zero point. It is a measuring scheme used in roads, rail, pipelines, tunnels, canals etc.
Setting Out Horizontal Curves on Site

The importance of a centreline on site is that it provides a reference line from which other feature such as channels, verges, tops and bottoms of embankments etc can be located from. Thus it is important that:

The centreline is set out and marked (pegged) with a high degree of accuracy

The pegs are protected and marked in such a way that site traffic can clearly see them and avoid accidentally hitting them.

If a peg is disturbed it can easily be relocated with the same high degree of accuracy as before.

There are a number of different methods by which a centreline can be set out, all of which can be summarised in two categories:

Traditional methods – which involve working along the centreline itself using the straights, intersection points and tangent points for reference. The equipment used for these methods include, tapes and theodolites or total stations.
Coordinate methods – which use control networks as reference. These networks take the form of control points located on site some distance away from the centreline. For this method, theodolites, totals stations or GPS receivers can be used.

**Setting out Circular Curves by Traditional Methods:**

There are 3 methods by which pegs on the centreline of circular curves can be set out:

1. Tangential angle method
2. Offsets from the tangent lengths
3. Offsets from the long chord

When traditional methods are being used it is first necessary to locate the intersection and tangent points of a curve. This procedure is carried out as follows:
Locate the two straights AC and BD and define them with at least two pegs on the ground for each straight. Use nails in the tops of the pegs to define them precisely.

Set up a theodolite over the nail in a peg on one of the straights (say AC) and sight the nail in another peg on AC so that the theodolite is pointing in the direction of intersection point I.

Drive two additional pegs x and y on AC such that straight BD will intersect the line xy. Again use nails in x and y for precision.

Join the nails in the tops of pegs x and y using a string line.

Move the theodolite and set it up over a peg on BD, then sight the other peg on BD so that the telescope is again pointing at I.

Fix the position of I by driving a peg where the line of sight from the theodolite on BD intersects the string line xy.

Move the theodolite to I and measure the angle AIB. Calculate the deflection angle $\theta$, from $\theta = 180^\circ - \text{angle AIB}$. 

Calculate the tangent lengths IT and IU using \([R \tan (\theta/2)]\). Fix points T and U by measuring back along the straights from I.

Check the setting out angle ITU which should equal to \((\theta/2)\).

*Locating I, T and U with two Instruments*

In some cases it may not be possible to locate a theodolite on point I due to inaccessibility of some kind. In this case the location of I and the tangent points can be carried out using two theodolites. The procedure in this case is as follows:
Choose two points A and B somewhere on the straights such that it is possible to sight A to B and B to A and also to measure AB.

Measure AB

Measure the angles $\alpha$ and $\beta$, calculate $\lambda$ from $\lambda = 180 - (\alpha + \beta)$ and obtain $\theta$ from $\theta = (\alpha + \beta)$.

Use the sine rule to calculate $IA$ and $IB$

Calculate the tangent lengths $IT$ and $IU$ using $[R \tan (\theta/2)]$.

Using $AT = IA - IT$ and $BU = IB - IU$ set out T from A and U from B.

If possible check that $ITU$ is equal to $(\theta/2)$
Setting Out Using Tangential Angles

This is the most accurate method of the traditional methods for setting out curves. It can be done using a theodolite and tape, two theodolites or a total station and pole reflector.

The formula used to determine the tangential angles is derived as follows:
The length of a chord $TK$ is given by:

$$TK = R2\alpha_1$$

where alpha is in radian

Converting to degrees:

$$\alpha_1 = \left(\frac{TK}{2R}\right)\left(\frac{180}{\pi}\right) \text{ degrees}$$

Similarly

$$\alpha_2 = \left(\frac{KL}{2R}\right)\left(\frac{180}{\pi}\right) \text{ degrees} \quad \text{and} \quad \alpha_3 = \left(\frac{LM}{2R}\right)\left(\frac{180}{\pi}\right) \text{ degrees}$$

In general

$$\alpha = \left(\frac{\text{chord length}}{R}\right)\left(\frac{90}{\pi}\right) \text{ degrees}$$
Using a Theodolite and tape
In this method a theodolite is set up at the tangent point and used to turn the tangential angles to define the directions to each centre line peg. The exact positions of these pegs are fixed by measuring with a tape from peg to peg in sequence. The exact calculation and setting out procedures are as follows:

Setting out Procedure:
Using the methods described previously, the tangent points are fixed and the theodolite is set up at one of them.

The intersection point I is sighted and the horizontal circle is set to read zero. The theodolite is rotated so that the tangential angle $\alpha_1$ for the first chord TK is set on the horizontal circle.
The first chord TK is then set out by lining in the tape with the theodolite along this direction and marking off the length of the chord from the tangent point. The chord lengths derived in the calculations are in the horizontal plane and therefore any slope on the ground must be accounted for. Once the first position is located it is marked with a peg and nail to define the exact location of K.
The telescope is then turned until the horizontal circle is set to equal $\alpha_1 + \alpha_2$ in the direction TL. With the end of the tape hooked over the nail in peg K, the length of the second chord KL is ‘swung’ to intersect TL. The point L is then pegged as before accounting for slope corrections if required.

This procedure is repeated for all other points on the curve until point U is reached. Then the theodolite is moved to point U and the tangential angle IUT is measured, which should equal $\theta/2$.

Example:
It is required to connect two intersecting straights whose deflection angle is 13°16’00” by a circular curve of radius 600m. The through chainage of the intersection point is 2745.72m and pegs are required on the centreline of the curve at exact multiples of 25m of through chainage.

Tabulate the data necessary to set out the curve by the tangential angles method using a theodolite and tape.
Solution:

\[ \text{Tangent length} = IT = R \tan \left( \frac{\theta}{2} \right) = 600 \tan 06'38'' = 69.78\text{m} \]

*Through chainage of T = Chainage of I – IT = 2745.72 – 69.78 = 2675.94m*

To fix the first point on the curve at 2700m (the next multiple of 25m) the initial sub chord is:

Initial sub-chord = 2700 – 2675.94 = 24.06m
Length of circular curve = LC = R\theta \quad (\theta \text{ in radians}).

\[ L_C = \left( 600 \times 13.2667 \times \pi \right) \parallel 180 = 138.93m \]

Through chainage of U = Chainage of T + LC = 2675.94 + 138.93 = 2814.87m

Hence a final sub-chord is also required since 25m chords can on be used up to 2800m

Length of final sub-chord = 2814.87 – 2800 = 14.87m

Hence the 3 chords necessary are:

Initial sub-chord = 24.06m \quad = \frac{90}{\pi} \times \left( \frac{24.06}{600} \right) = 01^\circ 08'56''

4 General chords = 25.00m \quad = \frac{90}{\pi} \times \left( \frac{25.00}{600} \right) = 01^\circ 11'37''

Final sub-chord = 14.87m \quad = \frac{90}{\pi} \times \left( \frac{14.87}{600} \right) = 00^\circ 42'36''
<table>
<thead>
<tr>
<th>Point</th>
<th>Chainage (m)</th>
<th>Chord length (m)</th>
<th>Individual tangential angle</th>
<th>Cumulative tangential angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>2675.94</td>
<td>0</td>
<td>00 00 00</td>
<td>00 00 00</td>
</tr>
<tr>
<td>C₁</td>
<td>2700.00</td>
<td>24.06</td>
<td>01 08 56 (\alpha_1)</td>
<td>01 08 56</td>
</tr>
<tr>
<td>C₂</td>
<td>2725.00</td>
<td>25.00</td>
<td>01 11 37 (\alpha_2)</td>
<td>02 20 33</td>
</tr>
<tr>
<td>C₃</td>
<td>2750.00</td>
<td>25.00</td>
<td>01 11 37 (\alpha_3)</td>
<td>03 32 10</td>
</tr>
<tr>
<td>C₄</td>
<td>2775.00</td>
<td>25.00</td>
<td>01 11 37 (\alpha_4)</td>
<td>04 43 47</td>
</tr>
<tr>
<td>C₅</td>
<td>2800.00</td>
<td>25.00</td>
<td>01 11 37 (\alpha_5)</td>
<td>05 55 24</td>
</tr>
<tr>
<td>U</td>
<td>2814.87</td>
<td>14.87</td>
<td>00 42 36 (\alpha_6)</td>
<td>06 38 00</td>
</tr>
</tbody>
</table>

\[\Sigma 138.93\] (checks) \[\theta/2 = 06^\circ 38'00''\] (checks)
Tangential Angles Method using a Total Station and Pole Mounted Reflector

In this method a total station is set up at the tangent point and used to turn the tangential angles as for the theodolite and tape method. However instead of measuring the chord lengths from peg to peg using the tape, the distance measurement component of the Total Station is used to measure the length to each peg directly from the tangent point.
The line $TK = \text{long chord of the curve are } TK = 2R \sin(\alpha_1)$

The line $TL = \text{long chord of the curve are } TL = 2R \sin(\alpha_1 + \alpha_2)$

The line $TM = \text{long chord of the curve are } TM = 2R \sin(\alpha_1 + \alpha_2 + \alpha_3)$

**Example:**
Using the same information from the previous example Tabulate the necessary data to set out the curve using a total station and reflector.

**Solution:**
The tangential angles, sub-chords lengths and chord lengths are calculated exactly as for the previous example. The Long chords are obtained as follows:

\[ TC_1 = 2R \sin \alpha_1 = 1200 \sin 01^\circ08'56'' = 24.06m \]

(same as initial sub-chord in previous example)

\[ TC_2 = 2R \sin(\alpha_1 + \alpha_2) = 1200 \sin 02^\circ20'33'' = 49.05m \]

\[ TC_3 = 2R \sin(\alpha_1 + \alpha_2 + \alpha_3) = 1200 \sin 03^\circ32'10'' = 74.01m \]

<table>
<thead>
<tr>
<th>Point being set out</th>
<th>Cumulative tangential angle to be turned from T relative to the line TI</th>
<th>Long chord (m) to be set out from T</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>00°00'00&quot;</td>
<td>0</td>
</tr>
<tr>
<td>C_1</td>
<td>01°08'56&quot;</td>
<td>24.06</td>
</tr>
<tr>
<td>C_2</td>
<td>02°20'33&quot;</td>
<td>49.05</td>
</tr>
<tr>
<td>C_3</td>
<td>03°32'10&quot;</td>
<td>74.01</td>
</tr>
<tr>
<td>C_4</td>
<td>04°43'47&quot;</td>
<td>98.95</td>
</tr>
<tr>
<td>C_5</td>
<td>05°55'24&quot;</td>
<td>123.84</td>
</tr>
<tr>
<td>U</td>
<td>06°38'00&quot;</td>
<td>138.62</td>
</tr>
</tbody>
</table>
Setting out using Offsets from Tangent Lengths

This method of setting out requires two steel tapes. It is suitable for short curves or small radius.

Require length of the offset $X$ and known distances along the tangent length $Y$ from $T$:

$$R^2 = (R - X)^2 + Y^2 \quad \text{and} \quad X = R - \sqrt{R^2 - Y^2}$$

This can be rewritten as: \( X = \frac{Y^2}{2R} \)
Setting out using Offsets from the Long Chord
This traditional method also requires two tapes and is also suitable for curves of small radius such as kerbs or boundary walls.
We require the offset length \( X \) from the chord TU at a distance \( Y \) from F. This may be obtained from the following formula:

\[
X = \sqrt{R^2 - Y^2} - \sqrt{R^2 - \left(\frac{W}{2}\right)^2}
\]

**Setting out Circular Curves by Coordinate Methods**

Coordinate methods are used nowadays in preference to the traditional methods as they are more efficient methods of setting out. Setting out of horizontal curves using coordinate methods can be done using either *Intersection; bearing and distance* or *GPS* methods.

C\(_1\) and C\(_2\) fixed by intersection from control points A and B

C\(_3\) and C\(_4\) fixed by bearing and distance from control point P
Calculating the coordinate points on the centre line

In all cases it is first necessary to obtain the coordinate of the points that are to be set out on the curve centre line.
P and Q above are ground control points and the coordinates of points C1, C2, C3, C4, C5 and C6 are required:

1. To begin the calculations the coordinates of I, T and U are required

2. Then the curve is designed as if it were to be set out using a total station (as described previously). Thus all the tangential angles ($\alpha$ values) and the long chord lengths are calculated (TC1, TC2,..TU).

3. From the coordinates of T and I calculate the Bearing of TI and use it with the tangential angles to calculate the bearings of all the long chords from T.

4. Using the bearings of the long chords together with their lengths calculate their coordinates from those of point T.
Deriving the Setting Out Data

When setting out curves using total stations or GPS the centreline coordinates will usually have been pre-computed and the coordinate file for the curve uploaded to the total station or GPS receiver. In such a case the operator does not have to do any calculations.

When setting out by theodolite and tape, intersection or bearing and distance methods are used. This is carried out according to the following procedure:

Knowing the coordinates of P, Q, C1, C2, C3, C4, C5 and C6, derive the bearings and horizontal lengths from P and Q to each point using rectangular to polar conversion.

Set out the pegs by either intersection from P and Q using bearings PC1 and QC1, PC2 and QC2 etc. Or else by bearing and distance from P and Q using the bearings and horizontal lengths from P to each point and then checking them using the bearings and horizontal lengths from Q.
Example:

The circular curve in the previous two examples is to be set out by intersection from two nearby traverse station A and B. The whole circle bearing of straight TI is obtained from the design as 63°27’14”, as are the coordinates of the entry tangent point T.

A  829.17mE  724.43mN
B  915.73mE  691.77mN
T  798.32mE  666.29mN
Using the relevant data from the previous examples calculate:

1. The coordinates of all the points on the centre line of the curve, which lie at exact 25m multiples of through chainage

2. The bearing AB and the bearings from A required to set out the points

3. The bearing BA and the bearings from B required to set out the points.

Solution
The coordinates of C1 (with ref to diagram above):

Bearing TC1 = bearing at TI + α1
    = 63°27′14″ + 01°08′56″ (from previous example)
    = 64°36′10″

Horizontal length TC1 = 24.06m (from previous examples)
ΔETC1 = 24.06 sin 64°36′10″ = +21.735m

ΔNTC1 = 24.06 cos 64°36′10″ = +10.319m

EC1 = ET + ΔETC1 = 820.055m

NC1 = NT + ΔNTC1 = 676.609m
The coordinates of C2

Bearing TC2 = bearing at TI + (α1 + α2)
   = 63°27’14” + 02°20’33” (from previous example)
   = 65°47’47”

Horizontal length TC2 = 49.05m (from previous examples)

ΔETC2 = 49.05 sin 65°47’47” = +44.738m

ΔNTC2 = 49.05 cos 65°47’47” = +20.110m

EC2 = ET + ΔETC2 = 843.058m

NC2 = NT + ΔNTC2 = 686.400m

The coordinates of C3, C4 and C5
The remaining points are calculated by repeating the above procedure giving:

C3 = 866.442mE, 695.220mN

C4 = 890.183mE, 703.063mN

C5 = 914.224mE, 709.908mN

Coordinates of U

The coordinates of U are calculated twice to provide a check. Firstly, they are calculated by repeating the procedures used to calculate the coordinates of points C1 to C5.

The values obtained are;

U = 928.652mE, 713.502mN
Secondly they are calculated by working along the straights from T to I to U as follows:

Bearing TI = 63°27’14”
Horizontal length TI = 69.78m

Hence
\[ \Delta ETI = 69.78 \sin 63°27’14” = +62.423m \]
\[ \Delta NTI = 69.78 \cos 63°27’14” = +31.186m \]
\[ EI = ET + \Delta ETI = 798.32 + 62.423 = 860.743m \]
\[ NI = NT + \Delta NTI = 666.29 + 31.186 = 697.476m \]

The angle of deflection \( \theta \) was 13°16’00”
Bearing IU = Bearing TI + \( \theta \) = 63°27’14” + 13°16’00” = 76°43’14”
Horizontal length IU = 69.78m

Therefore
\[ \Delta EIU = 69.78 \sin 76°43’14” = +67.914m \]
\[ \Delta NIU = 69.78 \cos 76°43’14” = +16.029m \]
\[ EU = EI + \Delta EIU = 928.657m \]
\[ NU = NI + \Delta NIU = 713.505m \]
These check to with a few millimetres of the values obtained for U along the long chord TU.
The bearings from $AB$ and $BA$ and the bearings from $A$ or $B$ to each point are calculated from the quadrants method of using rectangular to polar conversion as discussed previously:

<table>
<thead>
<tr>
<th>Point</th>
<th>Chainage (m)</th>
<th>Coordinates</th>
<th>Bearing from A</th>
<th>Bearing from B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>mE</td>
<td>mN</td>
<td>°</td>
</tr>
<tr>
<td>T</td>
<td>2675.94</td>
<td>798.32</td>
<td>666.29</td>
<td>207</td>
</tr>
<tr>
<td>$C_1$</td>
<td>2700.00</td>
<td>820.05(5)</td>
<td>676.61</td>
<td>190</td>
</tr>
<tr>
<td>$C_2$</td>
<td>2725.00</td>
<td>843.06</td>
<td>686.40</td>
<td>159</td>
</tr>
<tr>
<td>$C_3$</td>
<td>2750.00</td>
<td>866.44</td>
<td>695.22</td>
<td>128</td>
</tr>
<tr>
<td>$C_4$</td>
<td>2775.00</td>
<td>890.18</td>
<td>703.06</td>
<td>109</td>
</tr>
<tr>
<td>$C_5$</td>
<td>2800.00</td>
<td>914.22</td>
<td>709.91</td>
<td>99</td>
</tr>
<tr>
<td>U</td>
<td>2814.87</td>
<td>928.65</td>
<td>713.50</td>
<td>96</td>
</tr>
</tbody>
</table>

*Bearing $AB = 110°40'19"$*  
*Bearing $BA = 290°40'19"$*
**Quadrant I**
Bearings 0–90°
\[ \theta = \tan^{-1} \frac{\Delta E}{\Delta N} \]

**Quadrant II**
Bearings 90–180°
\[ \theta = \tan^{-1} \frac{\Delta E}{\Delta N} + 180° \]

**Quadrant III**
Bearings 180–270°
\[ \theta = \tan^{-1} \frac{\Delta E}{\Delta N} + 180° \]

**Quadrant IV**
Bearings 270–360°
\[ \theta = \tan^{-1} \frac{\Delta E}{\Delta N} + 360° \]