CHAPTER 3. INTRODUCTION TO MATRIX METHODS FOR STRUCTURAL ANALYSIS

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DEFINITIONS

Modern methods of structural analysis overcome some of the drawbacks of classical techniques. Statically determinate and indeterminate structures are solved the same way. Trusses, beams or frames are solved with the same method. The method can be automated so we can use computers to solve.

Structural analysis using the matrix method does not involve any new concepts of structural engineering or mechanics.

All linear elastic structures (statically determinate or indeterminate) are governed by systems of linear equations. Below are some important definitions needed for creating the system of equations.

Elements: The individual members that comprise a structure.

**Truss Element:** A member subjected to axial forces only.

**Beam Element:** A member that resists transverse forces and moments.

**Frame Element:** A member that resists moments and axial and transverse forces
**Node:** Nodes are locations in the structure where elements are connected.

**Degrees Of Freedom (DOF):** A possible displacement (translational or rotational)

**Stiffness:** Force required to induce a unit deformation in an elastic material

**Force:** Force along translational or rotational degrees of freedom

**Coordinates**

**Global Coordinates:** Identify (with a unique number assigned) each DOF of the structure in the global coordinates.

- Coordinates are assigned to each DOF for each node
- Coordinates govern entire structure, not individual members
- Global coordinates have a unique number for each DOF of the entire structure

**Local Coordinates:** Identify DOF at the element level
For example, local coordinates 4, 5, 6 of the frame element correspond to global coordinates 11, 10, 12 of the frame.

**Hooke’s Law:**
Hooke’s Law relates force to deformation of a linearly elastic material:

- of a linearly elastic member: \( \sigma = E \varepsilon \)
- or of an entire structure: \( \{f\} = [K]\{x\} \)

where \( \{f\} \) is a force vector that contains all of the forces that are acting on all the DOF (or coordinates); \( \{x\} \) is a vector that contains all of the displacements of all the structural DOF; \( [K] \) is the stiffness matrix that relates the two. Note that the stiffness matrix is an \( nxn \) matrix where \( n \) is the number of DOF. In what follows, we will create the stiffness matrix of a simple spring, a series of springs and a truss.
CREATING A STIFFNESS MATRIX

The form of a 4x4 stiffness matrix (for a structure with four DOF) is shown below. Our task is to find the coefficients $K_{ij}$ of our structure.

$$
\begin{array}{cccc}
  j = 1 & j = 2 & j = 3 & j = 4 \\
  i = 1 & K_{11} & K_{12} & \ldots & . \\
  i = 2 & K_{21} & K_{22} & . & . \\
  i = 3 & . & . & . & . \\
  i = 4 & . & . & . & K_{ij}
\end{array}
$$

Definition 1:
The stiffness coefficient $K_{ij}$ is the force at coordinate $i$ when a unit displacement is imposed at coordinate $j$, and the displacements at all other coordinates are constrained to zero (fixed).

Example 1. Creating the Stiffness matrix of a spring

Consider the spring in the figure. The spring constant, $k$, is the force required to cause a unit displacement: $f = kx$. Here, a force of 10 lb is needed to cause a displacement of 1in. At the same time, a reaction of 10 lbs develops at node 1 to keep the system in static equilibrium and ensures that the displacement of node 1 is zero.
If we consider *Definition 1*, we see that we defined $K_{22}$ as a force at coordinate 2 when a unit displacement is imposed at coordinate 2. Therefore, $K_{22}$ has a value of 10. We also defined $K_{12}$ as the force that develops at coordinate 1 when a unit displacement is imposed at coordinate 2. Therefore, $K_{12}$ has the value of −10. These values are inserted into the Stiffness Matrix for the element as follows:

$$K = \begin{bmatrix}
10 & -10 \\
-10 & 10
\end{bmatrix}$$

As seen, node 2 was used to fill column two of the stiffness matrix. Now, to fill in column one, we will work with the release of node 1 while fixing node 2.

A 10-lb force is applied at node 1, imposing a unit displacement on node 1. This force is equivalent to $K_{11}$. The reaction force of -10 lbs that develops at Node 1 is equivalent to $K_{21}$. Completing the remainder of the stiffness matrix, we obtain:

$$K = \begin{bmatrix}
10 & -10 \\
-10 & 10
\end{bmatrix}$$

Note that the stiffness of the bar is 10 k/in. As a generalization, the stiffness matrix, $K$, can be rewritten as:

$$K = 10 \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}$$

In other words, the stiffness matrix for a single spring element of stiffness $k$ can be written as:

$$K = k \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix}$$
Now assume that the spring is loaded as shown below. We can solve for the unknown displacement and force using the relationship $\{f\} = [K] \{x\}$, where $[K]$ is given above.

\[
\begin{pmatrix}
200 \\
F_2
\end{pmatrix} = \begin{pmatrix}
10 & -10 \\
-10 & 10
\end{pmatrix} \begin{pmatrix}
x_1 \\
0
\end{pmatrix}
\]

Solving the system of equations, we get:

- $x_1 = 20$ in
- $F_2 = -200$ lb
Example 2. Creating the Stiffness matrix of a series of springs

Let’s investigate a system made up of three springs.

\[
\begin{align*}
K_{11} -10 -20 &= 0; \quad K_{11} = 30 \\
K_{21} + 20 &= 0; \quad K_{21} = -20 \\
K_{31} + 10 &= 0; \quad K_{31} = -10
\end{align*}
\]

Namely:
\[
\begin{align*}
\Sigma F_{\text{at node1}}: \quad K_{11} -10 -20 &= 0; \quad K_{11} = 30 \\
\Sigma F_{\text{at node2}}: \quad K_{21} + 20 &= 0; \quad K_{21} = -20 \\
\Sigma F_{\text{at node3}}: \quad K_{31} + 10 &= 0; \quad K_{31} = -10
\end{align*}
\]
The first column is written as:

\[
\begin{bmatrix}
30 & 0 & 0 & 0 \\
-20 & 0 & 0 & 0 \\
-10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Repeat the calculations after fixing all nodes except node 2 to find the second column. Namely:
\[\Sigma F@1: \quad K_{12} + 20 = 0; \quad K_{12} = -20\]
\[\Sigma F@2: \quad K_{22} - 30 - 40 = 0; \quad K_{22} = 50\]
\[\Sigma F@4: \quad K_{42} + 30 = 0; \quad K_{32} = -30\]

Substituting into the stiffness matrix, we obtain:

\[
\begin{bmatrix}
30 & -20 & 0 & 0 \\
-20 & 50 & 0 & 0 \\
-10 & 0 & 0 & 0 \\
0 & -30 & 0 & 0
\end{bmatrix}
\]

Repeat the calculations after fixing all nodes except node 3 to find the third column. Namely:
\[K_{13} = -10\]
\[K_{23} = 0\]
\[K_{33} = 10\]
\[K_{43} = 0\]
Repeat the calculations after fixing all nodes except node 4 to find the fourth column.

\[
\begin{bmatrix}
30 & -20 & -10 & 0 \\
-20 & 50 & 0 & 0 \\
-10 & 0 & 10 & 0 \\
0 & -30 & 0 & 0
\end{bmatrix}
\]

Repeat the calculations after fixing all nodes except node 4 to find the fourth column. Namely:
\[K_{14} = 0\]
\[K_{24} = -30\]
\[K_{34} = 0\]
\[K_{44} = 30\]
Substituting into the Stiffness Matrix:

\[ K = \begin{bmatrix}
30 & -20 & -10 & 0 \\
-20 & 50 & 0 & -30 \\
-10 & 0 & 10 & 0 \\
0 & -30 & 0 & 30 \\
\end{bmatrix} \]

Now assume that the spring is loaded as shown below. We can solve for the unknown displacement and force using the relationship \( \{f\} = [K]\{x\} \), where \([K]\) is given above.

Solution:

**Known:**
- \( x_3 = 0 \)
- \( x_4 = 0 \)
- \( F_1 = 40 \text{ lb} \)
- \( F_2 = 100 \text{ lb} \)

**Unknown:**
- \( x_1 = ? \)
- \( x_2 = ? \)
- \( F_3 = ? \)
- \( F_4 = ? \)

Using the basic equation \( \{F\} = [K]\{x\} \):

\[
\begin{bmatrix}
40 \\
100 \\
F3 \\
F4 \\
\end{bmatrix} = \begin{bmatrix}
30 & -20 & -10 & 0 \\
-20 & 50 & 0 & -30 \\
-10 & 0 & 10 & 0 \\
0 & -30 & 0 & 30 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
0 \\
0 \\
\end{bmatrix}
\]

This matrix equation cannot be solved as is stands. It must divided into parts where only forces or displacements are the only unknowns. Also, since \( x_3 \) and \( x_4 \) are equal to zero, columns 3 and 4 can be eliminated from the equations.

\[
\begin{bmatrix}
40 \\
100 \\
F3 \\
F4 \\
\end{bmatrix} = \begin{bmatrix}
30 & -20 & -10 & 0 \\
-20 & 50 & 0 & -30 \\
-10 & 0 & 10 & 0 \\
0 & -30 & 0 & 30 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
0 \\
0 \\
\end{bmatrix}
\]
This leaves us with the reduced matrix equation:

\[
\begin{pmatrix} 40 \\ 100 \end{pmatrix} = \begin{pmatrix} 30 & -20 \\ -20 & 50 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

\[x_1 = 3.6364 \text{ in}\]
\[x_2 = 3.4545 \text{ in}\]

Once all the displacements are solved, they can be used to solve for the remaining forces.

\[
\begin{pmatrix} F_3 \\ F_4 \end{pmatrix} = \begin{pmatrix} -10 & 0 \\ 0 & -30 \end{pmatrix} \begin{pmatrix} 3.6363 \\ 3.4545 \end{pmatrix}
\]

\[F_3 = -36.3636 \text{ lb}\]
\[F_4 = -103.6364 \text{ lb}\]

**Example 3. Creating the Stiffness matrix of a series of springs using Strain Energy**

Let’s write the expression for the energy of this system:

The strain energy \(U\) can be expressed as:

\[
U = \frac{1}{2} k_1 (x_1-x_3)^2 + \frac{1}{2} k_2 (x_2-x_1)^2 + \frac{1}{2} k_3 (x_4-x_2)^2
\]

where \(x_i\) is the displacement of every node \(i\). The potential energy \(P\) can be expressed as:

\[
P = F_1 x_1 + F_2 x_2 + F_3 x_3 + F_4 x_4
\]

The total energy \(E\) in the system at any time is \(U-P\) In view of the equations above, this relationship can be written as:

\[
E(x) = \frac{1}{2} x^T K x - x^T F
\]
where $K = \begin{bmatrix} k_1 + k_2 & -k_2 & -k_1 & 0 \\ -k_2 & k_2 + k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix}$

After the loading $F$ is applied to the system, the system deforms. In this case, each DOF $i$ displaces an amount $x_i$. We can solve for these displacements by solving for the vector $x$ that minimizes the energy of the system. Taking the derivative of the energy with respect to $x$ and setting this derivative to zero:

$$\frac{\partial}{\partial x} \left[ \frac{1}{2} x^T K x - x^T F \right] = 0$$

we arrive at:

$$Kx = F$$

The solution of this equation for $x$ gives the displacement that each DOF has undergone to help the system achieve equilibrium. In other words, under loading, the system will deform to a configuration that minimizes the total energy of the system. This is its equilibrium position.
Homework 6:

1) Repeat the lecture on energy minimization. Solve for all with your MATLAB (do not turn this in)

2) Find the stiffness matrix K using the methods of example 2

3) Given that $k_1=10$ and $k_2=30\text{lb/in}$ for the spring below.
   - Create the stiffness matrix of the structure using the method of Example 2
   - Write the kinetic and potential energy expressions for the springs and create the stiffness matrix for the structure using the energy method.
   - Given that a force $F_1=-30$, is applied on the first node solve for the equilibrium position.
   - Plot the energy of the system in MATLAB.
   - Show graphically that the vector $x$ that minimizes the energy of the system is the same as the one that solves the equations $Kx=F$
   - Create an optimization function that minimizes the energy and finds the equilibrium position