MANE 4240 & CIVL 4240
Introduction to Finite Elements

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Introduction to 3D Elasticity
Reading assignment:

Appendix C+ 6.1+ 9.1 + Lecture notes

Summary:

• 3D elasticity problem
  • Governing differential equation + boundary conditions
  • Strain-displacement relationship
  • Stress-strain relationship
• Special cases
  2D (plane stress, plane strain)
  Axisymmetric body with axisymmetric loading
• Principle of minimum potential energy
1D Elasticity (axially loaded bar)

A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young’s modulus
u(x) = displacement of the bar at x

1. **Strong formulation**: Equilibrium equation + boundary conditions

**Equilibrium equation**
\[
\frac{d\sigma}{dx} + b = 0; \quad 0 < x < L
\]

**Boundary conditions**
\[
u = 0 \quad \text{at} \quad x = 0
\]
\[
EA \frac{du}{dx} = F \quad \text{at} \quad x = L
\]
2. Strain-displacement relationship: $\varepsilon(x) = \frac{du}{dx}$

3. Stress-strain (constitutive) relation: $\sigma(x) = E \varepsilon(x)$

E: Elastic (Young’s) modulus of bar
Problem definition

3D Elasticity

V: Volume of body
S: Total surface of the body

The deformation at point \( \mathbf{x} = [x, y, z]^T \)

is given by the 3 components of its displacement

\[ \mathbf{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \]

**NOTE:** \( \mathbf{u} = \mathbf{u}(x, y, z) \), i.e., each displacement component is a function of position
3D Elasticity:
EXTERNAL FORCES ACTING ON THE BODY

Two basic types of **external forces** act on a body

1. **Body force** (force per unit **volume**) e.g., weight, inertia, etc
2. **Surface traction** (force per unit **surface area**) e.g., friction
Body force: distributed force per unit volume (e.g., weight, inertia, etc)

$$X = \begin{bmatrix} X_a \\ X_b \\ X_c \end{bmatrix}$$

**NOTE:** If the body is accelerating, then the inertia force

$$\rho \vec{\ddot{u}} = \begin{bmatrix} \rho \dddot{u} \\ \rho \dot{\dddot{v}} \\ \rho \dddot{w} \end{bmatrix}$$

may be considered as part of $\vec{X}$

$$\vec{X} = \vec{\tilde{X}} - \rho \vec{\dddot{u}}$$
**Surface Traction**

**Traction:** Distributed force per unit surface area

Volume element $dV$

Volume $(V)$

\[
\mathbf{T}_S = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}
\]
If I take out a chunk of material from the body, I will see that, due to the external forces applied to it, there are reaction forces (e.g., due to the loads applied to a truss structure, internal forces develop in each truss member). For the cube in the figure, the **internal reaction forces per unit area** *(red arrows)* , on each surface, may be decomposed into three orthogonal components.
3D Elasticity

\( \sigma_x, \sigma_y \) and \( \sigma_z \) are normal stresses. The rest 6 are the shear stresses. Convention \( \tau_{xy} \) is the stress on the face perpendicular to the x-axis and points in the +ve y direction. Total of 9 stress components of which only 6 are independent since \( \tau_{xy} = \tau_{yx} \)

\[ \tau_{yz} = \tau_{zy} \]
\[ \tau_{zx} = \tau_{xz} \]

The stress vector is therefore

\[ \sigma = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yx} \\ \tau_{xz} \end{pmatrix} \]
Strains: 6 independent **strain components**

\[ \varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \]

Consider the equilibrium of a differential volume element to obtain the 3 **equilibrium equations** of elasticity

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_a = 0 \]

\[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + X_b = 0 \]

\[ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + X_c = 0 \]
Compactly;

\[
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}
\end{bmatrix}
\]

where

\[
\partial^T \sigma + X = 0
\]

(1)
3D elasticity problem is completely defined once we understand the following three concepts

- **Strong formulation** (governing differential equation + boundary conditions)
- **Strain-displacement relationship**
- **Stress-strain relationship**
1. **Strong formulation of the 3D elasticity problem:** “Given the externally applied loads (on $S_T$ and in $V$) and the specified displacements (on $S_u$) we want to solve for the resultant displacements, strains and stresses required to maintain equilibrium of the body.”
Equilibrium equations

\[ \partial^T \sigma + X = 0 \quad \text{in} \quad V \]  

(1)

Boundary conditions

1. **Displacement boundary conditions**: Displacements are specified on portion \( S_u \) of the boundary

\[ u = u^{\text{specified}} \quad \text{on} \quad S_u \]

2. **Traction (force) boundary conditions**: **Traction**s are specified on portion \( S_T \) of the boundary

Now, how do I express this mathematically?
Volume element \( dV \)

**Traction:** Distributed force per unit area

\[
\bar{T}_S = \begin{bmatrix}
p_x \\
p_y \\
p_z
\end{bmatrix}
\]
**Traction:** Distributed force per unit area

\[
T_S = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}
\]

If the unit outward normal to \(S_T\) : 
\[
n = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}
\]

Then
\[
\begin{align*}
p_x &= \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\
p_y &= \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z \\
p_z &= \tau_{xz} n_x + \tau_{zy} n_y + \sigma_z n_z
\end{align*}
\]
Consider the equilibrium of the wedge in x-direction

\[ p_x \, ds = \sigma_x \, dy + \tau_{xy} \, dx \]

\[ \Rightarrow p_x = \sigma_x \frac{dy}{ds} + \tau_{xy} \frac{dx}{ds} \]

\[ \Rightarrow p_x = \sigma_x n_x + \tau_{xy} n_y \]

Similarly

\[ p_y = \tau_{xy} n_x + \sigma_y n_y \]
3D elasticity problem is completely defined once we understand the following three concepts

- **Strong formulation** (governing differential equation + boundary conditions)

- **Strain-displacement relationship**

- **Stress-strain relationship**
2. Strain-displacement relationships:

\[ \varepsilon_x = \frac{\partial u}{\partial x} \]

\[ \varepsilon_y = \frac{\partial v}{\partial y} \]

\[ \varepsilon_z = \frac{\partial w}{\partial z} \]

\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]

\[ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \]

\[ \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \]
Compactly; \[ \mathbf{\varepsilon} = \partial \mathbf{u} \] 

\[ \mathbf{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \quad \partial = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \quad \mathbf{u} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \]
In 2D

\[ \varepsilon_x = \frac{A'B' - AB}{AB} = \frac{\left( dx + \left( u + \frac{\partial u}{\partial x} \cdot dx \right) - u \right) - dx}{dx} = \frac{\partial u}{\partial x} \]

\[ \varepsilon_y = \frac{A'C' - AC}{AC} = \frac{\left( dy + \left( v + \frac{\partial v}{\partial y} \cdot dy \right) - v \right) - dy}{dy} = \frac{\partial v}{\partial y} \]

\[ \gamma_{xy} = \frac{\pi}{2} - \text{angle} \quad (C' \quad A' \quad B') = \beta_1 + \beta_2 \approx \tan \beta_1 + \tan \beta_2 \]

\[ \approx \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \]
3D elasticity problem is completely defined once we understand the following three concepts

- Strong formulation (governing differential equation + boundary conditions)
- Strain-displacement relationship
- Stress-strain relationship
3. Stress-Strain relationship:

Linear elastic material (Hooke’s Law)

\[ \sigma = D \varepsilon \]  

(3)

Linear elastic isotropic material

\[
D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}
\end{bmatrix}
\]
Special cases:

1. **1D elastic bar** (only 1 component of the stress (stress) is nonzero. All other stress (strain) components are zero) Recall the (1) equilibrium, (2) strain-displacement and (3) stress-strain laws

2. **2D elastic problems:** 2 situations
   - PLANE STRESS
   - PLANE STRAIN

3. **3D elastic problem:** special case-axisymmetric body with axisymmetric loading (we will skip this)
PLANE STRESS: Only the in-plane stress components are nonzero

Nonzero stress components $\sigma_x, \sigma_y, \tau_{xy}$

Assumptions:
1. $h<<D$
2. Top and bottom surfaces are free from traction
3. $X_c=0$ and $p_z=0$
**PLANE STRESS** Examples:

1. Thin plate with a hole

![Diagram of thin plate with a hole](image.png)

2. Thin cantilever plate

![Diagram of thin cantilever plate](image.png)
**PLANE STRESS**

Nonzero **stresses**: $\sigma_x, \sigma_y, \tau_{xy}$

Nonzero **strains**: $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}$

Isotropic linear elastic stress-strain law $\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$

$\varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y)$

Hence, the $D$ matrix for the **plane stress case** is

$$D = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$
PLANE STRAIN: Only the in-plane strain components are nonzero

Nonzero strain components $\varepsilon_x, \varepsilon_y, \gamma_{xy}$

Assumptions:
1. Displacement components $u,v$ functions of $(x,y)$ only and $w=0$
2. Top and bottom surfaces are fixed
3. $X_c=0$
4. $p_x$ and $p_y$ do not vary with $z$
PLANE STRAIN

Examples:

1. Dam

2. Long cylindrical pressure vessel subjected to internal/external pressure and constrained at the ends
PLANE STRAIN

Nonzero stress: \( \sigma_x, \sigma_y, \sigma_z, \tau_{xy} \)

Nonzero strain components: \( \varepsilon_x, \varepsilon_y, \gamma_{xy} \)

Isotropic linear elastic stress-strain law \( \sigma = D \varepsilon \)

\[
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & 0 \\
\nu & 1-\nu & 0 \\
0 & 0 & \frac{1-2\nu}{2}
\end{bmatrix}
\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{pmatrix}
\]

\( \sigma_z = \nu(\sigma_x + \sigma_y) \)

Hence, the \( D \) matrix for the plane strain case is

\[
D = \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & 0 \\
\nu & 1-\nu & 0 \\
0 & 0 & \frac{1-2\nu}{2}
\end{bmatrix}
\]
Example problem

The square block is in \textit{plane strain} and is subjected to the following strains

\begin{align*}
\varepsilon_x &= 2xy \\
\varepsilon_y &= 3xy^2 \\
\gamma_{xy} &= x^2 + y^3
\end{align*}

Compute the displacement field (i.e., displacement components \( u(x,y) \) and \( v(x,y) \)) within the block
Solution

Recall from definition

\[ \varepsilon_x = \frac{\partial u}{\partial x} = 2xy \quad (1) \]
\[ \varepsilon_y = \frac{\partial v}{\partial y} = 3xy^2 \quad (2) \]
\[ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = x^2 + y^3 \quad (3) \]

Integrating (1) and (2)

\[ u(x, y) = x^2y + C_1(y) \quad (4) \]
\[ v(x, y) = xy^3 + C_2(x) \quad (5) \]

Arbitrary function of ‘x’

Arbitrary function of ‘y’
Plug expressions in (4) and (5) into equation (3)

\[
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = x^2 + y^3 \quad (3)
\]

\[
\Rightarrow \frac{\partial}{\partial y} \left[ x^2 y + C_1(y) \right] + \frac{\partial}{\partial x} \left[ xy^3 + C_2(x) \right] = x^2 + y^3
\]

\[
\Rightarrow x^2 + \frac{\partial C_1(y)}{\partial y} + y^3 + \frac{\partial C_2(x)}{\partial x} = x^2 + y^3
\]

\[
\Rightarrow \frac{\partial C_1(y)}{\partial y} + \frac{\partial C_2(x)}{\partial x} = 0
\]

Function of ‘y’  Function of ‘x’
Hence

\[
\frac{\partial C_1(y)}{\partial y} = -\frac{\partial C_2(x)}{\partial x} = C \quad \text{(a constant)}
\]

Integrate to obtain

\[
C_1(y) = Cy + D_1 \quad \text{D}_1 \text{ and } D_2 \text{ are two constants of integration}
\]

\[
C_2(x) = Cx + D_2
\]

Plug these back into equations (4) and (5)

(4) \( u(x, y) = x^2 y + Cy + D_1 \)

(5) \( v(x, y) = xy^3 - Cx + D_2 \)

How to find \( C, D_1 \text{ and } D_2 \)?
Use the 3 **boundary conditions**

\[
\begin{align*}
   u(0,0) &= 0 \\
   v(0,0) &= 0 \\
   v(2,0) &= 0
\end{align*}
\]

To obtain

\[
\begin{align*}
   C &= 0 \\
   D_1 &= 0 \\
   D_2 &= 0
\end{align*}
\]

Hence the solution is

\[
\begin{align*}
   u(x, y) &= x^2 y \\
   v(x, y) &= xy^3
\end{align*}
\]
**Principle of Minimum Potential Energy**

**Definition:** For a linear elastic body subjected to body forces \( \mathbf{X} = [X_a, X_b, X_c]^T \) and surface tractions \( \mathbf{T}_s = [p_x, p_y, p_z]^T \), causing displacements \( \mathbf{u} = [u, v, w]^T \) and strains \( \varepsilon \) and stresses \( \sigma \), the potential energy \( \Pi \) is defined as the strain energy minus the potential energy of the loads involving \( \mathbf{X} \) and \( \mathbf{T}_s \)

\[
\Pi = U - W
\]
\[
U = \frac{1}{2} \int_V \mathbf{\sigma}^T \mathbf{\varepsilon} \, dV
\]

\[
W = \int_V \mathbf{u}^T \mathbf{X} \, dV + \int_{S_T} \mathbf{u}^T \mathbf{T}_S \, dS
\]
**Strain energy of the elastic body**

Using the stress-strain law \( \sigma = D \varepsilon \)

\[
U = \frac{1}{2} \int_V \sigma^T \varepsilon \, dV = \frac{1}{2} \int_V \varepsilon^T D \varepsilon \, dV
\]

In 1D

\[
U = \frac{1}{2} \int_V \sigma \varepsilon \, dV = \frac{1}{2} \int_V E \varepsilon^2 \, dV = \frac{1}{2} \int_{x=0}^L E \varepsilon^2 \, Adx
\]

In 2D **plane stress** and **plane strain**

\[
U = \frac{1}{2} \int_V \left( \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy} \gamma_{xy} \right) \, dV
\]

Why?
**Principle of minimum potential energy:** Among all **admissible** displacement fields the one that satisfies the equilibrium equations also render the potential energy $\Pi$ a minimum.

“admissible displacement field”:
1. first derivative of the displacement components exist
2. satisfies the boundary conditions on $S_u$