Gradually Varied Flows: Mild & Steep Surface Water Profiles

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Introduction

1. Uniform flow theory is often used to size artificial channels (e.g., sewers).
2. Uniform flow serves as the standard reference for experimental and theoretical work (e.g., studies aimed at understanding dissipation and turbulent flow behavior in channels; stability to gravitational waves etc).

However, it needs to be emphasized that uniform flow (UF) in open channels is the exception rather than the rule. Factors that cause the flow to depart from being uniform include (i) irregularities in cross-section, alignment, roughness and slope of natural channels; (ii) man-made obstructions such as dams, bridge piers; (iii) control devices such as gates; and (iii) unsteadiness of flow caused by dynamic control structures and/or by time and spatial varying inputs and outputs such as runoff and infiltration. Even in the laboratory it is difficult, if not impossible, to produce a truly uniform flow because the length of the flume is often not sufficient to establish this flow regime.

The departure from uniform flow means that the flow velocity and depth vary from location to location along the channel. If flow non-uniformity occur over a relatively short distance, then the flow velocity and depth vary rapidly. That is, the rate of change of depth and velocity with distance is very large. This type of non-uniform flow is referred to as rapidly varying flow (RVF). Examples of RVF include hydraulic jumps and hydraulic bores. When the flow is a RVF, skin friction effects can frequently be neglected, and in many instances, solutions may be obtained by simultaneously considering the mass conservation law and the momentum equation. It must be stressed that momentum and energy are not equivalent concepts across hydraulic jumps and bores. Indeed, across hydraulic jumps and bores, momentum is conserved but energy is not.

On the other hand, if the flow varies gradually, skin friction resistance gives rise to surface profiles of considerable extent (sometimes called "Backwater Curves") whose quantitative description requires the integration of either the equations of continuity and momentum or continuity and energy. The fact that we can use the momentum or the energy implies equivalency of these two concepts for Gradually Varied Flow (GVF) problems. To be sure, most cases of non-uniform flow in open channels represent a combination of both RVF and GVF.

Types of Surface Water Profiles

Recalling that the differential form of the energy equation is:
\[
\frac{dE}{dx} = S_0 - S_f \Rightarrow \frac{d}{dx} \left( y + \frac{V^2}{2g} \right) = S_0 - S_f \Rightarrow \frac{d}{dx} \left( y + \frac{Q^2}{2gA^2} \right) = S_0 - S_f
\]

Or

\[
\frac{dy}{dx} + \frac{Q^2}{2g} \frac{d}{dy} \left( \frac{1}{A^2} \right) = S_0 - S_f \quad \Rightarrow \quad \frac{dy}{dx} + \frac{Q^2}{2g} \frac{d}{dy} \left( \frac{1}{A^2} \right) \frac{dy}{dx} = S_0 - S_f \quad \Rightarrow \quad \frac{dy}{dx} = \frac{S_0 - S_f}{1 + \frac{Q^2}{2g} \frac{d}{dy} \left( \frac{1}{A^2} \right)}
\]

Or

\[
\frac{dy}{dx} = \frac{S_0 - S_f}{1 + \frac{Q^2}{2g} \frac{d}{dy} \left( \frac{1}{A^2} \right)} = \frac{S_0 - S_f}{1 + \frac{Q^2}{2g} \left( -\frac{2A}{dy} \frac{dA}{dy} \right) A^4} = \frac{S_0 - S_f}{1 - \frac{Q^2T^2}{gA^3} - \frac{Q^2}{A^2} - \frac{1 - V^2}{c^2}}
\]

Therefore, the differential equation that governs 1-D steady flow in a channel is:

\[
\frac{dy}{dx} = \frac{S_0 - S_f}{1 - F^2}
\]

According to the above relation, the flow profile \( y(x) \) and its slope depends on the channel slope; frictional slope and the Froude number. The possible types profiles are now investigated.

**Mild Profiles (Denoted by M):**

The profile is said to be mild if the uniform flow depth is larger than the critical depth. That is, \( y_o > y_c \) implying that the uniform flow is subcritical.
**Example:** Water flows uniformly in a long and very wide river of width 72m towards a lake. The river's cross sectional geometry is approximately rectangular. The flow rate is $50 m^3/s$, the slope of the river is 0.000019 and its Manning roughness is 0.03. Is this flow mild?

**Solution:**

The flow rate per unit width is: $q = Q/B = 50/72 = 0.694 m^2/s$. Therefore, the critical depth is:

$$y_c = \left(\frac{q^2}{g}\right)^{1/3} = \left(\frac{0.694^2}{9.81}\right)^{1/3} = 0.366m$$

The uniform flow depth can be obtained from the Manning equation, namely,

$$Q = \frac{A}{n} R^{2/3} S_0^{1/2}$$

The hydraulic radius for a wide (i.e., $B > y$) rectangular channel is:

$$R = \frac{A}{P} = \frac{By}{B + 2y} = y$$

Therefore, the uniform depth is:

$$Q = \frac{A}{n} R^{2/3} S_0^{1/2} = \frac{A}{n} y_0^{2/3} S_0^{1/2} \Rightarrow 50 = \frac{50 y_0}{0.03} y_0^{2/3} (0.000019)^{1/2} \Rightarrow y_0 = 2.556m$$

This is a mild channel since $y_o = 2.556m > y_c = 0.366$

Also note that the Froude number associated with uniform flow conditions is:

$$F_{ro} = \frac{V_o}{\sqrt{gy_o}} = \frac{q/ y_o}{\sqrt{gy_o}} = \frac{0.694/2.556}{\sqrt{9.81\times2.556}} = 0.0173$$

That is, the uniform flow is subcritical. Note the very small value of the Froude number; this is typical of most rivers.

**Example:** Calculate the uniform flowrate in an earth-lined trapezoidal canal having bottom width of 3 m, sides sloping 1 vertical to 2 horizontal, laid in a slope of 0.0001and having a depth of 1.8 m. Is this a mild channel?

**Solution:**
The critical area and depth are:

\[ A_c = \left( \frac{Q^2 T}{g} \right)^{1/3} \Rightarrow 3y_c + 2 \times \frac{1}{2} y_c \times 2y_c = \left( \frac{4.93^2 \times (3 + 2 \times 2y_c)}{9.8} \right)^{1/3} \Rightarrow y_c = \]

This is a mild channel since \( y_o > y_c \).
Note that the water depth of M1 profile is larger than the original uniform depth and increases with distance. It is this increase in depth that forces the displacement of people to higher elevation at the upstream side of dams. This M1 profile is also called *backwater curve*.
M2-Profile:

\[ y_o > y > y_c \quad \rightarrow \quad S_f > S_o \quad \text{and} \quad F < 1 \quad \rightarrow \quad \frac{dh}{dx} < 0 \]
M3-Profile:

Steep Profiles (Denoted by S): 

The profile is said to be steep if the uniform flow depth is smaller than the critical depth. That is, \( y_o < y_c \).

Example: The gutter of a roadway is confined by a curb 0.15 m high. The cross-slope of the road is 30 to 1, as shown in the diagram below, and the roadway itself has a longitudinal slope of 1 to 100. A heavy rain creates a runoff discharge of 0.4 \( m^3/s \). Assuming that \( n=0.014 \), is this a mild or steep channel.
The critical area and depth are:

\[ A_c = \left( \frac{Q^2 T}{g} \right)^{1/3} \Rightarrow 15 y_c^2 = \left( \frac{0.4^2 \times 30 y_c}{9.8} \right)^{1/3} \Rightarrow y_c^{5/3} = \left( \frac{0.4^2 \times 30}{9.8} \right)^{1/3} / 15 \Rightarrow y_c = 0.17m \]

This is a steep channel since \( y_o < y_c \).
S1-Profile:

\[ y > y_c > y_o \]  \rightarrow  S_f < S_c < S_o \text{ and } F < 1  \rightarrow  \frac{dh}{dx} > 0

\[ \frac{dh}{dx} > 0 \]
$y_c > y > y_o \rightarrow S_f < S_o \text{ and } F > 1 \rightarrow \frac{dh}{dx} < 0$
S3-Profile:

\[
\frac{dy}{dx} > 0
\]

\[y_c > y_o > y \quad \Rightarrow \quad S_f > S_o \quad \text{and} \quad F > 1 \quad \Rightarrow \quad \frac{dh}{dx} > 0\]
There are 7 other profiles (please see Appendix).

**Examples:** Sketch the possible flow profiles in the channels shown in the following figures.

(a)

(b)
Computation of Flow Profiles:

The computation of the flow profile amounts to solving (integrating) the following governing equation of 1-D steady flow in a channel is:

\[
\frac{dy}{dx} = \frac{S_0 - S_f}{1 - F^2}
\]

Recall that the above is equivalent to:

\[
\frac{dE}{dx} = S_0 - S_f \Rightarrow \frac{d}{dx} \left( y + \frac{V^2}{2g} \right) = S_0 - S_f
\]
Integrating the above between station 1 and station 2 who are a distance \( \Delta x \) apart (see Figure below) gives:

\[
\int_1^2 d\left( y + \frac{V^2}{2g} \right) = \int_1^2 (S_0 - S_f) dx
\]

Or

\[
y_1 + \frac{V_1^2}{2g} + S_f \Delta x = y_2 + \frac{V_2^2}{2g} + S_0 \Delta x
\]

where \( S_0 \) is the channel slope and \( \overline{S_f} = \int_1^2 S_f dx = \text{average friction slope between station 1 and station 2.} \) The simplest way to obtain the average is:

\[
\overline{S_f} = \int_1^2 S_f dx \approx \frac{1}{2} (S_{f1} + S_{f2})
\]

Solving for \( \Delta x \) one gets:

\[
\Delta x = \frac{(y_1 - y_2) + (V_1^2 - V_2^2) / 2g}{S_0 - S_f} = \frac{(y_1 - y_2) + (V_1^2 - V_2^2) / 2g}{S_0 - \frac{1}{2} (S_{f1} + S_{f2})}
\]

The local friction slope can be obtained from the Chezy equation, the Darcy-Weisbach equation or the Manning equation. The Manning equation is often used for this purpose. As a result,

\[
S_f = \left( \frac{nV}{R^{2/3}} \right)^2
\]

Therefore:

\[
\Delta x = \frac{(y_1 - y_2) + (V_1^2 - V_2^2) / 2g}{S_0 - S_f} = \frac{(y_1 - y_2) + (V_1^2 - V_2^2) / 2g}{S_0 - \frac{1}{2} \left( \frac{n_1 V_1}{R_1^{2/3}} \right)^2 + \left( \frac{n_2 V_2}{R_2^{2/3}} \right)^2}
\]
Gradually varied flow (GVF) and velocity profile

The computation of the flow profile using Direct Step Method proceeds as follows.

1. Identify the control point (i.e., the mechanism responsible for establishing the flow profile such as a dam, a gate etc). Mathematically, this is called boundary condition.
2. The computation proceed in the upstream direction of the control point if the flow is subcritical. On the other hand, the computation proceed in the downstream direction of the control point if the flow is supercritical.
3. Let i be a counter that identifies a section whose coordinate is \( x_i \). Therefore, \( y_i, V_i, n_i, R_i, A_i, S_{fi}, E_i, P_i \) are the water depth, velocity, Manning coefficient, hydraulic radius, area, frictional slope, specific energy and wetted perimeter respectively.
4. The recursive formula to solve for the flow profile is:
\[
\Delta x = x_{i+1} - x_i = \frac{(y_{i+1} - y_i) + (V_{i+1}^2 - V_i^2) / 2g}{S_0 - \frac{1}{2} \left( \frac{n_i V_i}{R_i^{2/3}} \right)^2 + \left( \frac{n_{i+1} V_{i+1}}{R_{i+1}^{2/3}} \right)^2}
\]
5. Assuming that the Manning coefficient is constant and let \( i=1 \) at the control point. The flow rate \( Q \) is known and \( y_1, V_1, R_1, A_1, S_{f1}, E_1, P_1 \) at the control point are also known. Now, assume a value for the water depth at \( i=2 \). The assumed value, \( y_2 \) needs to be consistent with the flow profile. Using your assumed value of \( y_1 \) and the fact that you know the flowrate \( Q \) determine \( V_2, R_2, A_2, S_{f2}, E_2, P_2 \). Now, the
distance between station 0 and station 1 can be computed as follows:

\[
x_2 - x_1 \approx \frac{y_2 - y_1 + (V^2 - V^2_1) / 2g}{S_0 - \frac{1}{2} \left( \frac{n V_1}{R_2^{2/3}} \right) + \left( \frac{n V_2}{R_2^{2/3}} \right)^2} = \frac{E_2 - E_1}{S_0 - \frac{1}{2} \left( \frac{n V_1}{R_2^{2/3}} \right) + \left( \frac{n V_2}{R_2^{2/3}} \right)^2}
\]

6. Repeat step 5 for other values of \(i\).
7. At the end of your calculation you have the following pairs: \((y_1, x_1)\); \((y_2, x_2)\), \((y_3, x_3)\) etc. Using these pairs, one can obtain the flow profile by plotting \(y\) versus \(x\).

The computation just described is best performed in a tabular form as follows:

<table>
<thead>
<tr>
<th>(i)</th>
<th>(y_i)</th>
<th>(A_i)</th>
<th>(P_i)</th>
<th>(R_i^{2/3})</th>
<th>(V_i)</th>
<th>(V_i^2 / 2g)</th>
<th>(S_{fi})</th>
<th>((S_{fi} + S_{fi+1}) / 2)</th>
<th>(E_{i+1} - E_i)</th>
<th>(X_{i+1} - X_i)</th>
<th>(X_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Control Point</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>2</td>
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</tr>
</tbody>
</table>

**Example:** Water flows uniformly in a long and very wide river of width 72m towards a lake as shown in the figure below. In order to raise the water level of the lake by 0.6m, a dam will be constructed. The location of the dam is as indicated in the figure. The flow rate in the river is 50 m\(^3\)/s. This river has a slope of 0.000019 and a roughness, \(n\), of 0.03. Estimate the increase in depth at a station that is approximately 30 km upstream of the lake.

**Solution:**
The flow rate per unit width is: \( q = Q/B = 50/72 = 0.694 \text{ m}^2/\text{s} \). Therefore, the critical depth is:

\[
y_c = \left( \frac{q^2}{g} \right)^{1/3} = \left( \frac{0.694^2}{9.81} \right)^{1/3} = 0.366 \text{ m}
\]

The uniform flow depth can be obtained from the Manning equation as follows. As a result,

\[
Q = \frac{A}{n} R^{2/3} S_0^{1/2}
\]

The hydraulic radius for a wide (i.e., \( B \gg y \)) rectangular channel is:

\[
R = \frac{A}{P} = \frac{By}{B + 2y} = y
\]

Therefore, the uniform depth is:

\[
Q = \frac{A}{n} R^{2/3} S_0^{1/2} = \frac{A}{n} y_0^{2/3} S_0^{1/2} \implies 50 = \frac{50y_0}{0.03} y_0^{2/3} (0.000019)^{1/2} \implies y_0 = 2.556 \text{ m}
\]

This is a mild channel since \( y_o = 2.556 \text{ m} > y_c = 0.366 \). The flow profile is an M1 type (i.e., backwater curve).

The control point is station 1 (see figure below). This control point (CP) is established because of the presence of the dam.

The water depth at the control point is the original uniform depth plus the increment in depth induced by the dam. That is:

\[
y_1 = y_o + 0.6 = 3.156 \text{ m}
\]
At distance $\Delta x = -30.29$ Km upstream of the control point, the water depth is 2.94m. The negative sign means that station 2, 3 etc are upstream of the CP.

Assume that the depth is changing linearly from region $i=4$ to $i=5$, the depth at 30 Km upstream of the CP is:

$$2.95-0.01\times(30-28.58)/(30.29-28.58)=2.942\text{m}.$$